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MUTUAL COUPLING FROM FAR FIELD MEASUREMENTS.(U)
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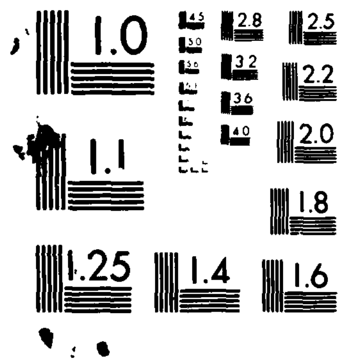

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"MUTUAL COUPLING FROM FAR FIELD MEASUREMENTS"

by

S. Chaiken
G. Franceschetti and
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U.S. Army Contract DAAG 29-79-C-0050

REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER / 2. GOVT ACCESSION NO. AD-A112 814	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Mutual Coupling From Far Field Measurements	5. TYPE OF REPORT & PERIOD COVERED
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) S. Chaiken, G. Franceschetti and N. G. Alexopoulos	8. CONTRACT OR GRANT NUMBER(s) DAAG 29-79-C-0050
9. PERFORMING ORGANIZATION NAME AND ADDRESS Electrical Engineering Department UCLA Los Angeles, CA 90024	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Dr. James Mink ARO - P 4800-C-81	12. REPORT DATE December 1981
	13. NUMBER OF PAGES
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Army Research Office Research Triangle Park North Carolina	15. SECURITY CLASS. (of this report) Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Mutual Coupling
Far Field Measurements

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A theory for mutual coupling between two dimensional canonical minimum scattering antennae is reviewed. Expressions for self and mutual impedance are formulated in terms of the measurable far field radiation diagram. The case of two identical aerials is considered in some detail. The constraint of bandlimitation is applied to the antenna pattern function to facilitate the required analytic continuation thereof. Numberical analytic continuation is addressed whereby sampling proximates and Miller-Tikhonov regularization

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Abstract

A theory for mutual coupling between two dimensional canonical minimum scattering antennae is reviewed. Expressions for self and mutual impedance are formulated in terms of the measurable far field radiation diagram. The case of two identical aerials is considered in some detail. The constraint of bandlimitation is applied to the antenna pattern function to facilitate the required analytic continuation thereof. Numerical analytic continuation is addressed whereby sampling proximates and Miller-Tikhonov regularization schemes are examined.

Consider a two dimensional, isotropic, canonical minimum scattering* antenna in a homogeneous, lossless medium set in the plane $y = 0$. This antenna produces a two dimensional E-type field i.e., $\underset{\sim}{E}$ parallel to the z-axis. (see figure 1.) Taking $\underset{\sim}{E}(x,y) = \underset{\sim}{E}(r,\phi)$ where $x = r\cos\phi$ and $y = r\sin\phi$, the spatial Fourier representation is

$$\underset{\sim}{E}(x,y) = E(x,y)\hat{z} = \int_{\mathbb{R}} \hat{\underset{\sim}{E}}(u)e^{-iux-ivy}du$$

$$= \int_{\mathbb{R}} \hat{E}(u)e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}du\ \hat{z}$$

where $\underset{\sim}{k} = u\hat{x} + v\hat{y}$, $\hat{k} = \underset{\sim}{k}/|\underset{\sim}{k}|$, $v = \begin{cases} \sqrt{\beta^2 - u^2} & |u| < \beta \\ -i\sqrt{u^2 - \beta^2} & |u| > \beta \end{cases}$ with

$\beta = 2\pi/\lambda$ and implicit time dependence, $e^{+i\omega t}$, assumed and suppressed. Note the choice of cuts for v is consistent with the restriction $y \geq 0$ for outwards traveling waves. On setting $u = \beta\cos\xi$, $v = \beta\sin\xi$, E transforms as

$$E(x,y) = E(r,\phi) = \beta \int_S \hat{E}(\xi)e^{-i\beta r\cos(\phi-\xi)} \cdot \sin\xi\, d\xi$$

where S is a Sommerfeld contour in the ξ-plane - see figure 2. Deforming S to a steepest descent path in the region of convergence for E gives the asymptotic

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*We use freely the previous study report [1] by Professor Franceschetti. Constraints as "invisibility", i.e., antennae that do not scatter with particular loadings of ports - "Canonical Minimum Scattering" being as such with ports open circuited - are discussed at great length therein.

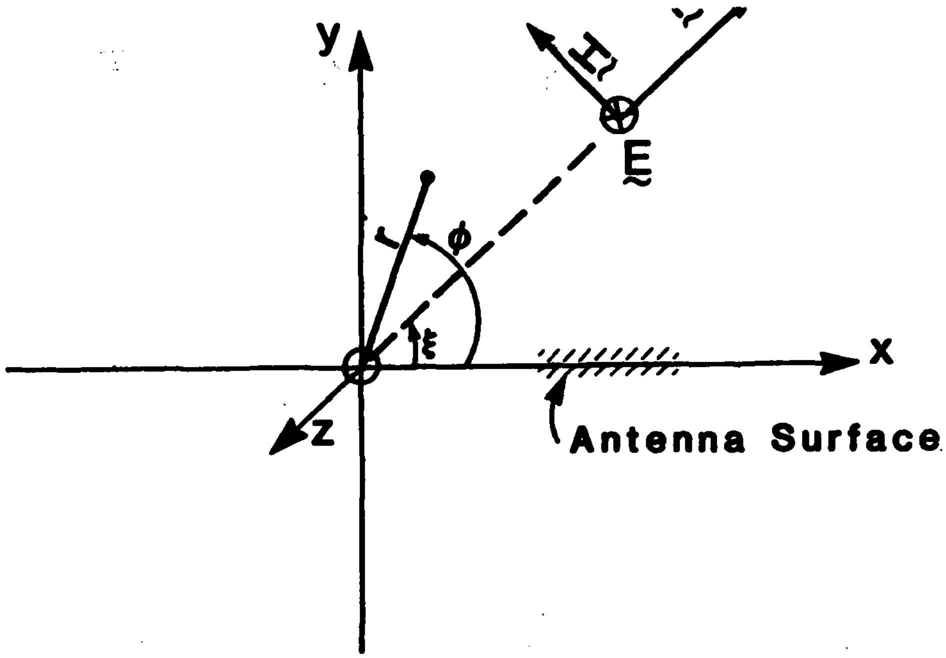

Figure 1 Antenna Geometry

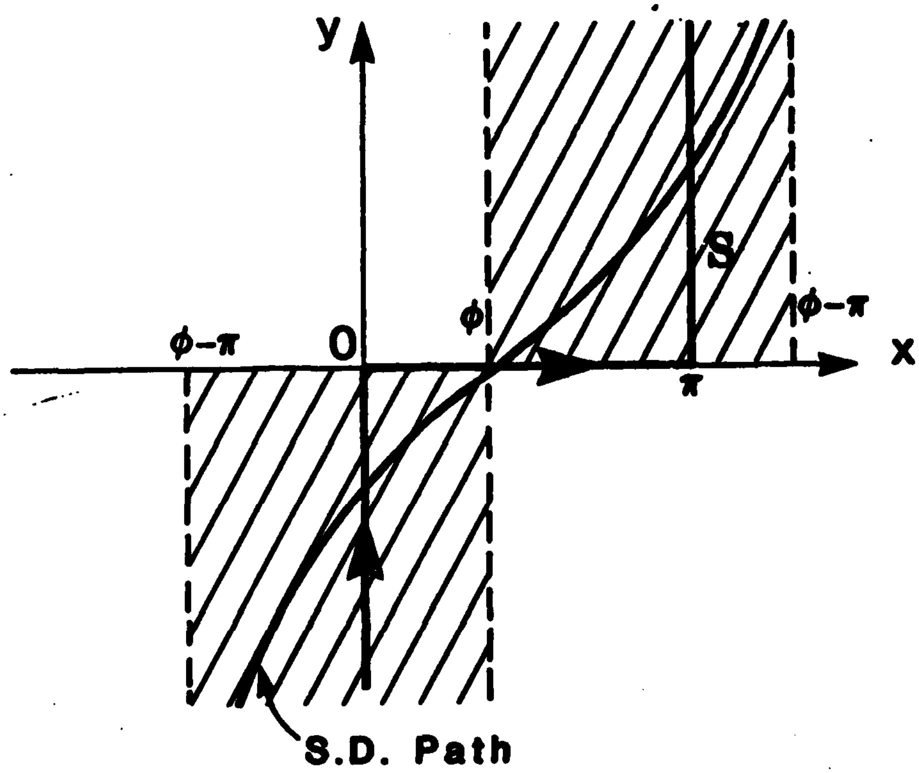

Figure 2 Sommerfeld Path

$$E(r >> 1,\phi) \doteq -\beta \sin \phi \, \hat{E}(\beta \cos \phi) \sqrt{\frac{2\pi}{-i b r}} \; e^{-i\beta r}$$

It is possible to define a normalized antenna current, I, such as to require [1] reciprocity, $f(\phi) = f_{rec}(\phi)$, where $f_{(rec)}$ is the transmitting (receiving) effective height per unit length. This imposes the following constraint on spectrum and pattern functions for real values of the observation angle $\phi \in [0,\pi]$

$$\hat{E}(\beta \cos \phi) = \frac{\beta f(\phi) \zeta I}{4\pi v}$$

The identity theorem provides for the coincidence of the above quantities throughout the common domains of analyticity. Thus, as $f(u) = f(\beta \cos \xi)$, we know the spectrum $\hat{E}(u)$ if we can find the analytic continuation of the pattern function, $f(u)$, from the visible region $u \in [-\beta,\beta]$ through all regions of common analyticity. As $f(u)$ & $\hat{E}(u)$ are often entire functions, this region is generally all of $\mathbb{C}$. Lastly, we note that the Fourier transform of the magnetic field associated with this E-polarized field, $\hat{\underline{H}}(u)$ has the following properties:

1) $\underline{H}(x,y) = H(x,y)\hat{z} = \int_{\mathbb{R}} \hat{H}(u)\hat{z} \, e^{-ik\cdot r} du$

2) $\zeta \, \hat{\underline{H}}(u) = \hat{z} \times \hat{\underline{E}}(u)$

With these results it is easy to find the total power, P_{tot}, injected into the region $y \geq 0$ by the antenna

$$E(x,0) = \int_{\mathbb{R}} \hat{E}(u) e^{-iux} du$$

$$\zeta H(x,0) = \int_{\mathbb{R}} (\hat{k} \times \hat{z}) \hat{E}(u) e^{-iux} du$$

$$P_{tot} = \frac{1}{2}\int_{\mathbb{R}} \underset{\sim}{E} \times \underset{\sim}{H}^{*}\cdot\hat{y}\ dx = \frac{1}{2}\frac{2\pi}{\omega\mu}\int_{\mathbb{R}} \hat{E}(u)\hat{E}^{*}(u)v\ du$$

$$= \frac{1}{2}\frac{2\pi}{\omega\mu}\frac{\beta^2}{16\pi^2}\zeta^2|I|^2\int_{\mathbb{R}}\frac{f(u)f^{*}(u)}{v}\ du$$

Frequently $P_{tot} = \frac{Z}{2}|I|^2$ where $Z = R + iX$, the antenna system input impedance(e.g. "one mode slots" in ground plane or infinite wires). Here one has a relation between a measurable quantity $Z(\omega)$ and $f(u)$ which we shall exploit presently. Note this constraint applies to both "visible" and "invisible" values of $f(u)$.

Finally we give a formal expression for the mutual coupling between antennae. (See Figure 3.) Reference lines of Antennae: I = (0,0), II = $(x_o,y_o) = (r_o,\phi_o)$. Take the plane-wave spectrum of antenna I with antenna II open circuited (note invisibility assumption for II), then the spectrum of the radiated field is:

$$\hat{E}_I(u) = \frac{\beta}{4\pi v} f_1(\xi)\zeta I_1 .$$

The open circuit voltage at antenna II due to the spectral component $E_I(u)du$ is

$$dV_{21} = \frac{\beta\zeta I_1 e^{-iux_o - ivy_o}}{4\pi v}\ du\ f_I(\xi)f_{II}(?)$$

Now for given ξ, $\hat{k} = \cos\xi\,\hat{x} + \sin\xi\,\hat{y}$ whereby the radiation impinges on II at angle $\xi + \pi$ (see Figure 3.) thus by superposition:

$$V_{21} = \frac{\beta\zeta I_1}{4\pi}\int_{\mathbb{R}}\frac{f_I(\xi)f_{II}(\xi+\pi)}{v}e^{-iux_o - ivy_o}du \qquad \text{or}$$

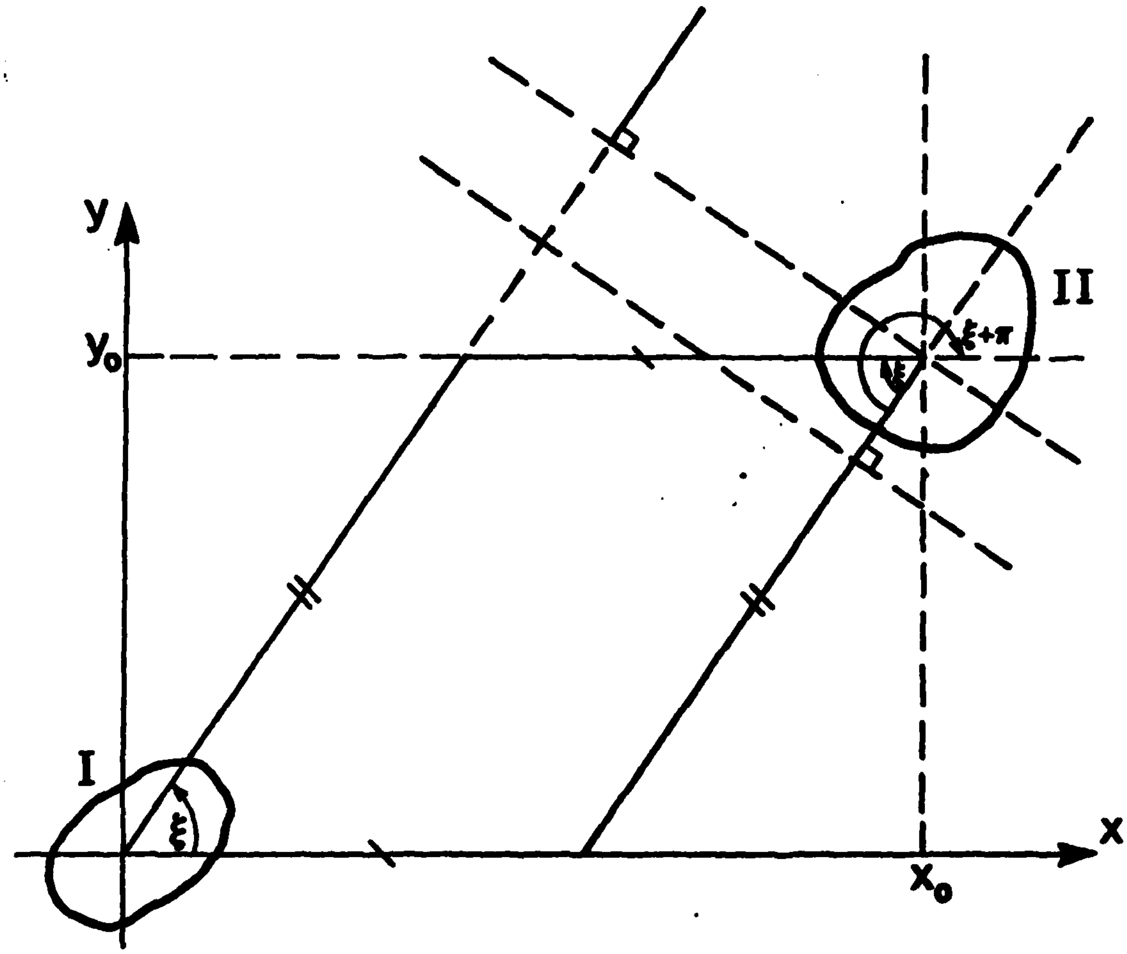

Figure 3 Mutual Coupling Geometry

$$Z_{21} = \frac{V_{21}}{I_I} = \text{mutual impedance between 2 C.M.S. } 2^d \text{ antennae.} = \int_{\mathbb{R}} \frac{\zeta\beta}{4\pi} \frac{f_I(\xi) f_{II}(\xi + \pi) e^{-iux_o - ivy_o}}{v} du$$

$$= \int_S f_1(\xi) f_2(\xi + \pi) e^{-i\beta r_o \cos(\phi_o - \xi)} d\xi$$

by the previous change of variable.

Now we assume the antennae to be identical (e.g. in an array of slot radiators etc.) whence these expressions simplify for real excitations:

$$\hat{E}(u) = \frac{1}{2\pi} \int_{\mathbb{R}} E(x,0) e^{-iux} dx, \text{ note } -u = \beta \cos(\xi + \pi)$$

if $u = \beta \cos \xi$ then $\hat{E}(-u) = \hat{E}(u)^*$ where $E(x,0)$ is the "antenna current" (equivalent) excitation at $y = 0$. Therefore

$$Z_{21} = \frac{\zeta\beta}{4\pi} \int_{\mathbb{R}} \frac{f(\xi) f^*(\xi) e^{-iux_o}}{v} du = \frac{\zeta\beta}{4\pi} \int_{\mathbb{R}} \frac{P(u) e^{-iux_o}}{\sqrt{\beta^2 - u^2}} du$$

where $P(u)$ is the conventional power density with $u = \beta \cos \xi$ and the antenna system, again, constrained to $y = 0$. If we force the sources to be of finite extent the pattern functions become band-limited i.e.

$$f(u) = f(\beta \cos \xi) = \int_{-a}^{a} e^{i(\beta \cos \xi) x} j(x) dx$$

where the excitation, $j(x)$, is nonzero $x \in [-a,a]$ ($\mathbf{j} = j(x)\delta(y)\hat{z}$ is an equivalent current.) The sequel considers the utility of this constraint of band-limitation and the possibility of relaxing it as well as the requirement of identicallity of radiators previously imposed. We should state here that we are attempting to calculate the mutual couplings between aerials by means of the previously deduced

integrals which require knowledge of the pattern or power pattern function over both visible and invisible values of argument. At hand are the values of the power and pattern function in the visible $[-\beta \leq u \leq \beta]$ region and possibly the input impedance over some range of frequencies. The required numerical analytic continuation to invisible values is a very delicate process (a point to be amplified below) and requires much constraint to provide practicability. In that one may approximate any entire function by a sequence of band-limited functions, the constraint of band-limited proximates to a given pattern function is not too onerous. While the general problems of approximation in this context will be considered extensively below, the virtue of this scheme will be displayed now.

Consider the power like quantity $f_I(u)f_{II}(-u)$. Note that given f_1, f_2 B_a;[*] The product quantity, $f_I(u)f_{II}(-u)$, is also band-limited as follows from the convolution theorem[†]. In fact, $f_I(u)f_{II}(-u) \in B_{2a}$. In the case of identical aerials this $f(u)f(-u) = P(u)$ - the conventional power pattern function, which may be measured with some accuracy thereby constituting a reasonable candidate for numerical analytic continuation. (The inaccuracy of phase measurement would seem to forbid an attempt at continuation of the pattern function itself.) Let D be the distance between aerials' reference lines then

$$Z_{21} = \frac{\zeta\beta}{4\pi} \int_{\mathbb{R}} \frac{P(u)e^{-iuD}du}{\sqrt{\beta^2 - u^2}}, \text{ if } \quad f(u) = \int_{-a}^{a} e^{-iux} g(x)dx$$

[*]Space of band-limited functions i.e. $f(t) = \int_{-a}^{a} e^{i\omega t} g(\omega)d\omega$ where $g(\omega)$ is an $L^2(-\tau, \tau)$ function i.e. "square integrable."

[†]$F_1(\omega)F_2(\omega) \leftrightarrow f_1(t)*f_2(t)$ where $f(t) = \int_{\mathbb{R}} e^{i\omega t}F(\omega)\frac{d\omega}{2\pi}$ etc.

Then $P(u) = f(u)\cdot f(-u) = |f(u)|^2 \leftrightarrow 2\pi\, [g(x)*g(-x)](x) = G(x)$

$x \in [-2a, 2a]$.

$= 0$ otherwise

Hence $P(u) = \int_{-2a}^{2a} G(x)\, e^{-iux}dx$. Expanding $G(x)$ in an exponential series in $[-2a, 2a]$; $G(x) = \sum_{N} C_n\, e^{in\pi x/2a}$ with $C_n = \frac{1}{4a}\int_{-2a}^{2a} G(x)\, e^{-in\pi x/2a}dx = \frac{1}{4a} P\left(\frac{n\pi}{2a}\right)$. So one gets the "sampling" theorem representation:

$$P(u) = \int_{-2a}^{2a} \sum_{N} C_n e^{-iux} e^{in\pi/2a} dx = \sum_{N} C_n \int_{-2a}^{2a} e^{ix\left(\frac{n\pi}{2a} - u\right)} dx$$

$$= \sum_{N} P\left(\frac{n\pi}{2a}\right) \frac{\mathrm{Sin}(2au-n\pi)}{2au-n\pi}$$

Now insert this result into the coupling integral and invoke the "convolution theorem"

$$\int_{-\infty}^{\infty} \frac{P(u)e^{-iuD}}{\sqrt{\beta^2-u^2}} du = \sum_{N} P\left(\frac{n\pi}{2a}\right) \int_{R} \frac{\mathrm{Sin}(2au-n\pi)}{(2au-n\pi)} \frac{e^{-iuD}}{\sqrt{\beta^2-u^2}} du$$

$$= \sum_{N} P\left(\frac{n\pi}{2a}\right) I(u,\beta,D)$$

Recall the following fourier transform pairs

1) $\frac{\mathrm{Sin}(2au-n\pi)}{2au-n\pi} \leftrightarrow \frac{e^{ixn\pi/2a}}{4a}$ $x \in [-2a, 2a]$

$= 0$ otherwise

2) and for our choice of cuts (cf. p. 1).

$$\frac{1}{\sqrt{\beta^2-u^2}} \longleftrightarrow \frac{H_o^{(2)}(\beta|x|)}{2} \quad \text{hence}$$

$$I\,(n,\beta,D) = \frac{\pi}{8a}\int_{-2a}^{2a} e^{in\pi x/2a}\, H_o^{(2)}\,(\beta|D+x|)\,dx.$$ Now recall further

the integral representation for $H_o^{(2)}$ over finite limits

$$H_o^{(2)}(z) = \frac{2}{\pi}\, i \int_0^{\pi/2} \frac{e^{-i(z+\theta/2)}e^{-2z\,\mathrm{Cot}\,\theta}}{\mathrm{Sin}\theta\,\sqrt{\cos\theta}}\,d\theta \quad \{\mathrm{Re}\ z > 0\}$$

This may be used in the expression for $I(n,\beta,D)$ and limits of integration exchanged to reduce $I(n,\beta,D)$ to a concatenation of elementary functions readily amenable to evaluation with 1) fortran complex algebra 2) any simple numerical integration scheme. e.g.

for $D > 2a$ we have

$$I(n,\beta,D) = 2ie^{-i\beta D}\int_0^{\pi/2} \frac{e^{-i\theta/2}\,e^{-2\beta D\,\mathrm{Cot}\,\theta}}{\sin\theta\,\sqrt{\cos\theta}}\;\frac{\mathrm{Sin}(n\pi-2a\beta+4\beta ia\,\mathrm{Cot}\,\theta)}{(n\pi-2a\beta+4ia\beta\,\mathrm{Cot}\,\theta)}\,d\theta$$

with a somewhat more tedious form for $D < 2a$.

So for reciprocal antennae with real excitation the first order (exact C.M.S.) mutual coupling Z_{12} may be expressed

$$Z_{21} = \frac{\zeta\beta}{4\pi}\int_{-\beta}^{\beta} \frac{P(u)}{\sqrt{\beta^2-u^2}}\,e^{-iuD}du = \frac{\zeta\beta}{4\pi}\sum P\left(\frac{n\pi}{2a}\right) I(n,\beta,D)$$ Note that

in the integral formulation the reactive part of the coupling is extracted from the integration over the invisible ($\beta < |u|$) range of the pattern. Likewise the radiative portion of the coupling from the visible portion, to see this note

$$Z_{21} = \frac{\zeta\beta}{4\pi} \int_{-\beta}^{\beta} \frac{P(u)e^{-iuD}}{\sqrt{\beta^2 - v^2}} du + \frac{i\zeta\beta}{4\pi} \int_{|u|>\beta} \frac{P(u)e^{-iuD}}{\sqrt{u^2 - \beta^2}} du$$

where $P(u)$ is in general an even function hence the first integral is obviously pure real as is the second. However, we have reduced this quadrature to a series whose n^{th} term has both real and imaginary parts. Thus the "sampling transformation" displays the contribution of both visible and invisible parts of the pattern to the total mutual impedance. It should be pointed out that $P(\frac{n\pi}{2a})$ and $I(n,\beta,D)$ are both "Fourier-series" coefficients of $G(x)$ and $H_o^{(2)}(\beta|D + x|)$, hence decay as $O(\frac{1}{n})$ and give a somewhat tardy convergence of $O(\frac{1}{n^{2+}})$ to the mutual impedance series. It is possible to improve this result somewhat as we shall show later. Now however, the problem at hand!

How can one extract the first few $P(\frac{n\pi}{2a})$ from the known $P(0)$ and $P(u)$ for $u \in [-\beta,\beta]$ and thereby estimate the coupling for two identical aerials? How may one analytically continue from the segment the bandlimited function $P(u)$? We should point out that, depending on the antenna structure(visible range - dimensions of antenna - $(\frac{-2a}{\lambda}, \frac{2a}{\lambda})$) $P(0)$ $P(\frac{\pm\pi}{2a})$ may be known. In fact for highly directive antennae, in which case the coupling question is somewhat moot, the extensive nature of the visible range contains sufficient sample points $(-\beta < \frac{\pm n\pi}{2a} < \beta)$ to yield a very accurate picture of the coupling. Exclusive thereof, we are faced with the solution of the so-called "ill-posed problem" of "numerical analytic continuation" of band-limited functions. The following discussion considers numerous naive but tractable solutions to this problem and at length some sophisticated yet awkward methods of solution. We shall require also some information of the sampling theorem and its cognates

to speed the convergences of the various series involved. We commence with a general discussion of numerical analytic continuation.

Analytic continuation, cornerstone of complex analysis, is not directly numerical in character. The usual context is that of a "magic wand" waved over a given functional representation known valid in a restricted region but convergent in a larger region. The agreement of these representations on any dense set (just a limit point!) provides for total agreement in any common domain of analyticity. Hence one "analytically continues", the function from some small set, with reckless abandon, throughout C or so it seems. In the case of an <u>exact</u> agreement, e.g. two integral representations of $\Gamma(z)$, analytic continuation is a deft tool. Numerical (experimental) attempts, however, are characterized by the following playful Theorems [2], used by Atkinson, quoted below.

Divine Theorem: If one knows an analytic function on a segment of a line, inside its domain of analyticity, one knows it throughout this domain of analyticity.

Diabolical Theorem: If the function is not known exactly on the line segment, but only within an error corridor of width 2ϵ, then the uncertainty in the continued function is such that its value at any given point, in the domain of analyticity, can be any number whatsoever, and this for any ϵ, no matter how small.

All seems, but is not, lost. It turns out that "global bounds" on the function to be continued [3] will restore the "continuous data dependence" lost in the process. However the restrictions suitable to a given problem are ad hoc. The sequel will examine two "almost-best-possible methods of analytic continuation based on the Miller-Tikhonov regularization schemes. This exposition is borrowed with modification from recent optics

literature of image extrapolation and object restoration. These methods, which utilize the eigenfunctions of the finite Fourier transform, the Prolate-Spheroidal wave functions, are quite cumbersome to employ. They may be used, eventually, as benchmarks to compare with proposed "sampling" proximates to be introduced shortly. It should be remarked that we are here concerned with the analytic continuation of band-limited functions i.e. given $f(u) = \int_{-a}^{a} e^{iux} g(x)dx$ <u>and</u> $f(u)$, $u \in [-\beta,\beta]$ find $f(u)$ for $u \notin [-\beta,\beta]$; or (harder) find $g(x) \forall x \in [-a,a]$ in the above Fredholm equation of the first kind, whence one may compute $f(u)$ from the transform relationship above. It is this duality between the analytic continuation of band-limited functions and the Fredholm equation that partially motivates our requirement of band-limitation. The general problem of analytic continuation, by means of the Cauchy theorem, can be expressed as the solution of a singular integral equation [3] which is an insidious numeric task, whereas, the constraint of band-limitation requires the "solution" to a regular integral equation of the first kind - a simpler task well considered in the literature. Let us consider a band-limited function $f(u)$ with $f(u)=\int_{-2a}^{2a} e^{iux}F(x)dx \in \mathbb{B}_{2a}$. First note [4] the sequence $\left\{\frac{\sin(2au-n\pi)}{2au-n\pi}\right\}_{\mathbb{N}}$ is a complete, orthogonal sequence in $\mathbb{R}$ all of whose terms are band-limited, $\in \mathbb{B}_{2a}$. If one were to consider the distribution due to various currents at $y = 0$ in the previous geometry with the insistence of piecewise constant currents (thus approximating a given smooth distribution) the pattern would be of this form. Recall we have previously the band-limited power pattern as a sampling series

$$P(u) = \sum_{\mathbb{N}} P(\tfrac{n\pi}{2a}) \frac{\sin(2au - n\pi)}{(2au - n\pi)}$$

We might ask the following:

1) Is it possible to approximate $P(u)$ in the visible range with a "sampling series" of unknown coefficients with the constraint of antenna self-impedance matched as measured?

2) Is it possible to solve for $P(\frac{n\pi}{2a})$ as a (truncated) set of linear equations that we implicitly define through the data as follows:

A. Record $P(u_i)$ for $\{u_i\}_{i=1}^{2N}$ and u_i $[-\beta,\beta]$ for all i.

B. Solve the truncated set of equations

$$4a\ P(u_i) = \sum_{N}^{N} P(\frac{jn}{2a}) \frac{\sin(2au_i - j\pi)}{2au_i - j\pi} \qquad i = 1,\ 2N$$

for $P(\frac{j\pi}{2a})$, $j = -N, N$. (recall $P(0)$ known)

C. Impose the impedance constraint on these near singular linear equations.

3) Is it possible to hasten the convergence of the "sampling series" and provide for solution of fewer coefficients, thus taking advantage of our ability to sample the visible interval at will? (e.g. find a derivative of $P(u)$)

All questions are answered in the affirmative, and involve novel techniques that circumvent the previously required usage of the Prolate Spheroidal Harmonics. We shall investigate each in turn.

1) First, one requires a criterion of fidelity. Without loss of generality, we take a mean-square $(L_2(-2a,2a))$ setting. Take $\tilde{P}(u) = \sum_{i=N}^{N} a_i \frac{\sin(2au - n\pi)}{(2au - n\pi)}$, with $a_o = P(0)$ known, as the sampling proximate. Find $\int_{-\beta}^{\beta} (\tilde{P}(u) - P(u))^2 du = \underset{\{a_i\}}{\text{minimum}}$, subject to the

constraint Z_{in} is given. Recall Z_{in} may be expressed in terms of the pattern function as previously shown. To wit

$$Z_{in} = R + iX = \frac{\zeta\beta}{8\pi} \int_{-\beta}^{\beta} \frac{\tilde{P}(u)}{\sqrt{\beta^2 - u^2}}\, du + \frac{\zeta\beta i}{8\pi} \int_{|u|>\beta} \frac{\tilde{P}(u)}{\sqrt{\beta^2 - u^2}}\, du$$

The quadratures may be found by direct integration of the above quantities or the previous convolution approach i.e.

$$Z_{in} = R + iX = \frac{\zeta\beta}{8\pi} \int_{\mathbb{R}} \frac{\tilde{P}(u)}{\sqrt{\beta^2 - u^2}}\, du = \frac{\zeta\beta}{8\pi} \sum_{N} a_n \int_{\mathbb{R}} \left(\frac{\sin(2u - n\pi)}{2au - n\pi}\right)\frac{du}{\sqrt{\beta^2 - u^2}}$$

$$= \frac{\zeta\beta}{8\pi} \sum_{n=-N}^{N} a_n I(n,\beta,0)$$

where

$$I(n,\beta,0) = \int_{-2a}^{2a} \underbrace{\frac{\pi}{4a}\, e^{-in\pi x/2a}}_{(\text{even}+\text{odd})}\; \underbrace{H_0^{(2)}(\beta|x|)}_{(\text{even})}\, dx$$

which, again, may be reduced to a 1^d integration of elementary functions by the prior integral representation for the Hankel function. We have <u>two</u> constraints on $\{a_n\}_{n=-N}^{N}$:

$$R = \frac{\zeta\beta}{8\pi} \sum_{n=-N}^{N} a_n \mathrm{Re}(I(n,\beta,0)) \qquad X = \frac{\zeta\beta}{8\pi} \sum_{n=-N}^{N} a_n \mathrm{Im}(I(n,\beta,0))$$

It is now a simple matter to use Lagrange multipliers to solve for $\{a_i\}$ such that $\int_{-\beta}^{\beta} (P(u) - P(u))^2 du = \underset{\{a_i\}}{\text{minimum}}$ given X,R. The functional of $\{a_i\}$ to be minimized:

$$J\{a_i\} = \int_{-\beta}^{\beta} \left(\sum_{-N}^{N} a_n \frac{\sin 2au - n\pi}{2a - n\pi} - P(u)\right)^2 du + \lambda_1 \sum_{-N}^{N} a_n I_n' + \lambda_2 \sum_{-N}^{N} a_n I_n''$$

where $I_n' = \text{Re}\, I(n,\beta,0)$ etc. and $\frac{8\pi}{\zeta\beta} R = R_o = \sum_{-N}^{N} a_n I_n'$,
$\frac{8\pi}{\zeta\beta} X = X_o = \sum_{-N}^{N} a_n I_n''$. $\frac{\partial J\{a_i\}}{\partial a_i} = 0$ for all a_i, $i = \pm 1, \pm 2, \ldots, \pm N$
requires

$$2 \sum_{n=-N}^{N} a_n \int_{-\beta}^{\beta} \frac{\sin 2au - n\pi}{2au - n\pi} \frac{\sin(2au - i\pi)}{(2au - u\pi)} dv - 2 \int_{-\beta}^{\beta} P(0) \frac{\sin(2au - i\pi)}{(2au - i\pi)} du$$

$$+ \lambda I_i' + \lambda_2 I_i'' = 0$$

The last $2N + 2$ equations may now be solved for the $2N + 2$ unknowns $\{a_i\}_{i=-N}^{N}$, λ_1, λ_2. To minimize sampling of $P(u)$ in evaluating the second integral one could use, say, a Gauss-Legendre quadrature. Note that the finite range and analytic nature of the integrands makes their evaluation a simple and inexpensive task. Results numeric of this procedure are contained in the first appendix.

2) The possibility of using the continuum of pattern values in the visible to extract a continuation through the sampling theorem is not new [5]. One may measure at will and solve the set of linear equations described above. Goodman's classic text on optics [6] considers this procedure and provides it blessings. In point of fact, one is apt to get order of magnitude results or better in its blind application. However, the linear set is not, to begin with, favorably conditioned and any attempt to continue "very far" will lead to trouble. The problem of constraints or regularization in this connection has not been explored in the literature, nor has the very specialized nature of the linear equations involved (to wit they have the form of the "double-alternant"). We intend to pursue these topics at length, and while results to date are incomplete, we give cause to further these activities. Part of the problem stems from the slowly convergent nature of the sampling series.

This we shall examine in some detail, offering "quicker" sampling proximates as means to the required coefficients. Lastly, we indicate how the transformed expressions for mutual coupling might be improved through this device.

Recall that we wish to solve

$$P(u_i) = \sum_{j=-N}^{N} P\left(\frac{j\pi}{2a}\right) \frac{\sin(2au_i - j\pi)}{2au_i - j\pi}, \qquad i = 1,2N$$

for $P(\frac{j\pi}{2a})$ $j = -N,\ldots,N, j \neq 0$ (which is known, $P(0)$) by accurate samplings of $P(u_i)$ for u_i $[-\beta,\beta]$ subject to the self-impedance constraints, truncated as $X_o = \sum_{-N}^{N} P(\frac{n\pi}{2a})I''_n$, $R_o = \sum_{-N}^{N} P(\frac{n\pi}{2a})I'_n$ with I''_n etc. as previously defined. For the moment we ignore these constraints and consider just the set of linear equations above. As previously remarked, direct solution thereof produces "trash" for N of any size. Out of curiosity, the least-squares (normal) solution was also considered for a few test functions (Appendix B, pt. I) i.e.

If $\underset{\approx}{A}\, \underset{\sim}{x} = \underset{\sim}{b}$ where $\underset{\sim}{b} \longleftrightarrow \{P(u_i)\}$; $\underset{\sim}{x} \longleftrightarrow \{P(\frac{j\pi}{2a})\}$ $\underset{\approx}{A} \longleftrightarrow \left(\frac{\sin(2au_i - j\pi)}{2au_i - j\pi}\right)$

the nonsingular least squares problem $\underset{\approx}{A}^T \underset{\approx}{A}\, \underset{\sim}{x}' = \underset{\approx}{A}^T \underset{\sim}{b}$ has $\underset{\sim}{x}'$ as solution to the problem $\| \underset{\approx}{A}x' - \underset{\sim}{b} \|_2 =$ minimum, uniquely. The results (Appendix B) indicate order of magnitude agreement "in trend" with the true coefficients. (This will be seen consistent with Tikhonov regularization of the illposed linear set.) Furthermore these truncated equations are of a peculiar form. Previous investigators [7,8] have considered similar sets and noticed closed form solutions may be readily obtained from Cramer's rule as such equations have the form of the double-alternant [9].

To see this we renormalize* the equations as follows:

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*assume $P(0)$ unknown to simplify indices.

$\sin(2au_i - j\pi) = [\sin 2au_i](-1)^j$, let $\tilde{P}(\frac{j\pi}{2a}) = (-1)^j P(\frac{j\pi}{2a}) = A_j$, $\frac{P(u_i)}{\sin 2au_i} = B_i$, $\gamma_j = [(j-1) - N]\pi$, $\delta_i = 2au_i$. Thus we obtain

$$B_i = \sum_{j=1}^{2N+1} A_j \frac{1}{\delta_i - \gamma_j} \qquad \begin{array}{l} i = 1, 2N+1 \\ j = 1, 2N+1 \end{array}$$

or

$$\underset{\sim}{B} = \begin{pmatrix} \frac{1}{\delta_1 - \gamma} & \cdots & \frac{1}{\delta_1 - \gamma_{2N+1}} \\ \vdots & & \vdots \\ \frac{1}{\delta_{2N+1} - \gamma_1} & \cdots & \frac{1}{\delta_{2N+1} - \gamma_{2N+1}} \end{pmatrix} \underset{\sim}{A} = \underset{\approx}{C}\, \underset{\sim}{A}$$

Now the determinant of C may be found in simple closed form. Some more notation [9] is required however.

Define the "difference-product" of the array $(a_1, a_2, \ldots a_n)$ as follows

$$\zeta^{1/2}(a_1, a_2, \ldots a_n) = \prod_{i+2}^{n} \prod_{j=1}^{i-1} (a_i - a_j)$$

then [9]

$$|\underset{\approx}{C}| = \mathrm{Det}|\underset{\approx}{C}| = \frac{(-1)^{(2N+1)N} \zeta^{1/2}(\delta_1, \delta_2, \ldots \delta_{2N+1}) \zeta^{1/2}(\gamma_1, \gamma_2, \ldots, \gamma_{2N+1})}{\prod_{K=1}^{2N+1} U_K}$$

where U_K is a cross product i.e. $U_K = \prod_{i=1}^{2N+1} (\delta_K - \gamma_i)$. Using Cramer's rule on $\underset{\sim}{B} = \underset{\approx}{C}\, \underset{\sim}{A}$ and taking $|\underset{\approx}{C}_{ij}|$ as the deletion of the ith column,

jth row of $|\underset{\approx}{C}|$ in the usual expansion about minors, gives the following explicit form for the solution of the truncated (infinite) set:

$$\underset{\sim}{B} = \underset{\approx}{C}\,\underset{\sim}{A} \Longrightarrow A_i = \sum_j (-1)^{j+1} \frac{|C_{ij}|}{|\underset{\approx}{C}|} \cdot B_j$$

where B_i are the (measured) r.h.s. of the normalized equation set allowing one thereby

1) to see how an error in $\{B_i\}$ might propagate.

2) the explicit calculation of the product solution and suggestive perhaps of a new analytic-product form representative of the band-limited function.

3) to settle the question of what, if any, constraints on $\{B_i\}$ are necessary to obtain the solution of these equations [e.g. 5] in the presence of noise or otherwise (Linfoot and Shephard [8] found in a similar but not applicable set explicit constraints on the various $\{B_i\}$, $\{\delta_i - \delta_{i+1}\}$, $\{\gamma_i - \gamma_{i+1}\}$ etc. dictating convergence.) We are presently attempting to resolve these questions and can provide here only a few initial observations.

As the row-column deletions leave yet another <u>alternant form</u> $\frac{|\underset{\approx}{C}_{ij}|}{|\underset{\approx}{C}|} = \Delta_{ji}$ is mildly tractable, just delete γ_i and δ_j from the expression for $|\underset{\approx}{C}|$ and divide obtaining

$$\Delta_{ji} = \frac{\prod_{K=1}^{2N+1} (\delta_K - \gamma_i) \prod_{K=1}^{2N+1} (\delta_j - \gamma_K) \cdot 1}{\pi^{2N} (i-1)!\,(2N+1-i)!\; (\delta_j - \gamma_i) \prod_{K=1}^{j-1} (\delta_j - \delta_K) \prod_{K=j+1}^{2N+1} (\delta_K - \delta_j)}$$

hence

$$A_i = \frac{\prod_{K=1}^{2N+1} (\delta_K - \gamma_i)(-1)^i}{\pi^{2N}(i-1)!(2N+1-i)!} \sum_{j=1}^{2N+1} \frac{(-)^j \prod_{K=1}^{2N+1} (\delta_j - \gamma_K) B_j}{(\delta_j - \gamma_i) \prod_{K=1}^{j-1} (\delta_j - \delta_K) \prod_{K=j+1}^{2N+1} (\delta_K - \delta_j)}$$

it is now necessary to assume some regularity on the sample points δ_K. If $\delta_{i+1} - \delta_i{}^* = \Delta$, for all i, the monster simplifies thus

$$A_i = \frac{(-)^i \prod_{k=1}^{2N+1} (\delta_k - \gamma_i)}{\Delta^{2N}\pi^{2N}(i-1)!(2N+1-i)} \sum_{j=1}^{2N+1} \frac{(-)^j B_j \prod_{k=1}^{2N+1} (\delta_j - \gamma_k)}{(\delta_j - \gamma_i)(j-1)!(2N+1-j)!}$$

One advantage here, the option of calculation in extended precision of the coefficients of B_j hence allowing one to "Monte Carlo" the direct solution and examine questions of behaviour.

The linear set of equations, previous, is known ill-conditioned. Intuitively, the compressed visible range, relative to the range of hopeful extension (say $|\beta| < u < N|\beta|$, $N = 5,7$) and the slow convergence of the sampling series tend to require a conditioning problem. One can't enlarge the visible range. Since we can sample freely however in its continuum, it is reasonable to ask if the "sampling" series rate of convergence may be enhanced. To this end we discuss briefly the convergence properties of the sampling series and produce the long awaited "quick" proximates promised previously. We then use these new series to obtain an analytic continuation of the power pattern function.

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*i.e., $\delta_i = \Delta i + \delta_0$

To start, consider the mean-square error in truncation of the sampling series over $\mathbb{R}$. Recall the orthogonality relationship of $\{\mathrm{sinc}(u-n)\}_{\mathbb{N}}^{*}$ on $\mathbb{R}$, and let

$$\epsilon_N(u) = f(u) - f_N(u) = \sum_{|n| \geq N+1} \frac{1}{\pi} f(n)\,\mathrm{sinc}(u-n)$$

where $f_N = \sum_{|n| \leq N} \frac{1}{\pi} f(n)\,\mathrm{sinc}(u-n)$ † and f is of exponential type π, bandlimited $[-\pi,\pi]$. The m.s.e.

$$E = \int_{-\infty}^{\infty} \epsilon_N \bar{\epsilon}_N du = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \sum_{\mathbb{N}} f(n)\,\mathrm{sinc}(u-n) \sum_{\mathbb{N}} \bar{f}(n')\,\mathrm{sinc}(u-n')du.$$

Note orthogonality kills the cross terms hence $E = \frac{1}{\pi^2} \sum_{|n| \geq N+1} |f(n)|^2$. The $\frac{f(n)}{\pi}$ are Fourier coefficients and decay at least as $O(\frac{1}{n})$. Thus the m.s.e. in truncation is (cf. [10], p. 170) with some work $\sim \frac{1}{N^2}$. As mean-square error is quadratic, one might expect and would find absolute error $\sim \frac{1}{N}$, hence the "tardy" convergence observed. Indeed, if $N >> 1$

$$\epsilon_1 = |f - f_N| = \Big|\sum_{|n| > N} \frac{f(n)}{\pi} \frac{\sin \pi t}{\pi(t-n)}\Big| \leq \sum_{|n| > N} \frac{|f(n)|}{|t-n|} \cdot \underbrace{\Big|\frac{\sin \pi t}{\pi^2}\Big|}_{\sim O(1)}$$

and

$$\sum_{n=N}^{\infty} \frac{|f(n)|}{|t-n|} \sim O\Big(\sum_{n=N}^{\infty} \frac{1}{n(n-t)}\Big) \sim \int_N^{\infty} \frac{du}{u(u-t)} = \frac{1}{N-t} \sim \epsilon_1 .$$

$^*\mathrm{sinc}\, x = \frac{\sin \pi x}{\pi x}$

†for convenience $2a = \pi$, renormalization.

Published results indicate [11, 12, 13] that though it is possible to "fine tune" the above, the magnitudes must stand. We now derive sampling proximates that display enhanced pointwise convergence of arbitrary order, given knowledge of f and its derivatives in the visible region, at arbitrary points interior to the sample points (unknowns , $F(\frac{n\pi}{2a})$).

Let $g(x)$ be of exponential type π (this scaling to simplify notation) so that

$$g(z) = \int_{-\pi}^{\pi} e^{izt}\phi(t)dt$$

where $\phi(t)$ is $L^2[-\pi,\pi]$. Form the quantity $\tilde{g}(z) = \frac{g(z) - g(0) - g'(0)z}{z^2}$ where without loss of generality we require knowledge of g, g', g'' (later at $z = 0$. One can readily generalize to any point in the visible region. Note $\tilde{g}(z)$ has a removable singularity at zero and

$$\lim_{z \to 0} \tilde{g}(z) = \lim_{z \to 0} \frac{g(0) + \frac{z}{1!}g'(0) + \frac{z^2}{2!} g''(0) + \cdots - g(0) - g'(0)\frac{z}{1!}}{z^2} = \frac{g''(0)}{2}$$

These quantities exist and will be $L^2(\mathbb{R})$ for $g(z)$ of modest growth i.e. $g(z) \sim O(z^\alpha)$ where $\alpha < 3/2$. In this case, $\tilde{g}(z)$ $L_2(\mathbb{R})$, so by the Paley-Wiener Theorem*, since $\tilde{g}$ is of exponential type π, it is band-limited $[-\pi,\pi]$. Now replace $\tilde{g}(z)$ by its Cardinal (Sampling) Series representation:

$$\tilde{g}(z) = \sum_{\mathbb{N}} \frac{(-)^n}{\pi(z-n)} \sin\pi z[\frac{g(n)}{n^2} - \frac{g(0)}{n^2} - \frac{g'(0)}{n}].$$

We require the following identities:

$$\text{(i)} \quad \sum_{\mathbb{N}}{}' \frac{(-)^n \sin\pi}{n\pi(z-n)} = \frac{1}{z} - \frac{\sin\pi z}{\pi z^2} = \frac{1}{z}\left(1 - \frac{\sin\pi z}{\pi z}\right)$$ where hereafter

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*The entire function $f(z)$ is of order 1, type c and belongs to $L^2(\mathbb{R})$ iff $f(z) = \frac{c}{2\pi}\int_{-1}^{1} e^{iczs}F(s)ds$ where $F(s)$ is $L^2(-1,1)$.

the prime will denote deletion of $n = 0$ term.)

Proof: Observe $\lim_{z \to 0} \frac{1}{z}\left(1 - \frac{\sin \pi z}{\pi z}\right) = \lim_{z \to 0} \frac{1}{z}\left(1 - 1 - \frac{(\pi z)^3}{3!}\right) = \lim_{z \to 0} \frac{\pi z^2}{3!} = 0,$

hence the quantity is entire. As $\operatorname{sinc}(z)$ is bounded,

$L^2(\mathbb{R})$, and of exponential type π, we have, again by the Paley-Wiener Th., the entire quantity is band-limited. Thus using again the sampling theorem

$$\frac{1}{z}\left(1 - \frac{\sin \pi z}{\pi z}\right) = \sum_{\mathbb{N}} \frac{1}{n} \underbrace{\left(1 - \frac{\sin \pi}{\pi z}\right)}_{= \delta(0,n)} \frac{(\sin \pi z)(-)^n}{\pi(z-n)} = \sum_{\mathbb{N}}{}' \frac{(-)^n \sin \pi z}{\pi n(z-n)} .$$

$$\text{(ii)} \sum{}' \frac{1}{n^2} \frac{\sin \pi z (-)^n}{\pi(z-n)} = \frac{1}{z^2}\left[1 - \frac{\sin \pi z}{\pi z}\right] - \frac{\pi^2 \sin \pi z}{3! \; \pi z}$$

Proof: Consider $\frac{1}{z^2}\left[1 - \frac{\sin \pi z}{\pi z}\right]$, note the quantity has a removable singularity at zero and is thus Entire, of type π. Further, the limiting value at zero is $\pi^2/3!$. As the quantity is also $L^2(\mathbb{R})$ the sampling theorem applies

$$\frac{1}{z^2}\left[1 - \frac{\sin \pi z}{\pi z}\right] = \sum_{\mathbb{N}}{}' \frac{1}{n^2}\left[1 - \frac{\sin n\pi}{\pi n}\right] \frac{\sin \pi z (-)^n}{\pi(z-n)} + \frac{\pi^2}{3!} \frac{\sin \pi z}{\pi z} .$$

These results set directly into the sampling expression for $\tilde{g}(z)$, holding in mind that $\tilde{g}(0) = g''(0)/2$ give:

$$\tilde{g}(z) = \frac{\sin \pi z}{2\pi z} g''(0) + \sum_{\mathbb{N}}{}' \frac{(-)^n \sin \pi z \; g(n)}{\pi n^2 (z - n)} - g'(0) \frac{1}{z}\left[1 - \frac{\sin \pi z}{\pi z}\right]$$

$$- g(0)\left[\frac{1}{z^2}\left(1 - \frac{\sin \pi z}{\pi z}\right) - \frac{\pi^2 \sin \pi z}{3! \; \pi z}\right]$$

Now, writing $\tilde{g}(z)$ in terms of $g(z)$ as previously defined gives

$$g(z) = \frac{z\sin\pi z}{2\pi} g''(0) + g'(0)\frac{\sin\pi z}{\pi} + g(0)\left[\frac{\sin\pi z}{n\pi} + \frac{z\pi\sin\pi z}{3!}\right]$$

$$+ \sum_{N}{}' \frac{(-)^n \cdot \sin\pi z \cdot g(n) \cdot z^2}{n^2 \cdot \pi \cdot (z-n)}$$

The result is confirmed* [14] in the literature. We note that knowledge of higher derivatives at a point in the visible region produces a "faster" family of series [15]. Also, estimating the (second) derivative in the presence of noise is difficult enough - again an "ill-posed" problem, but a well studied one. (e.g. splines, interpolations etc. [16]). To obtain these even "faster" estimators one need only substitute more terms of the Taylor series cf. $\tilde{g}(z)$. Late literature quotes a "slower" result requiring knowledge of only one derivative [17] - the proof does not generalize:

$$g(z) = \frac{\sin z}{\pi} g'(0) + \frac{g(0)\sin\pi z}{\pi z} + \sum_{N}{}' \frac{(-)^n \sin\pi z \cdot g(n) \cdot z}{\pi n(z-n)}$$

Somewhat less ambitious but more practicable. At this juncture it would be possible to establish error bounds - a little thought, however, would convince

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*not shown, general discussion [15] gives germ for proof.

the reader that the previous bounds are improved by a factor of n in the series denominators so results are one to two magnitudes better.

We illustrate for $\phi(t) = |\sin t|$ the explicit series form. (A numerical table of convergences and error is in Appendix B, pt. II). Note $|\sin t|$ has the required vanishing properties at $\pm\pi$ and is not too smooth.

$$g(z) = \int_{-\pi}^{\pi} e^{izt} |\sin t| dt = \frac{2(1 + \cos\pi z)}{1 - z^2}, \qquad g(0) = 4$$

$$g(n) = 0 \qquad n \text{ odd}$$

$$g(n) = 4/(1 - n^2) \qquad n \text{ even.}$$

A simple way to find derivatives is the direct differentiation of the integral representation. Whereupon, $g'(0) = 0$, $g''(0) = 8 - 2\pi^2$.

Thus the three series interpolations for $g(z)$

1) $$g(z) = \sum_{\text{even } n} \frac{\sin\pi z}{\pi(z-n)} \left(\frac{4}{1 - n^2}\right) \qquad \text{(Cardinal Series)}$$

2) $$g(z) = \frac{4\sin\pi z}{\pi z} + \sum_{\text{even } n}{}' \frac{(\sin\pi z) \cdot z\left(\frac{4}{1 - n^2}\right)}{\pi n(z - n)}$$

3) $$g(z) = [8 - 2\pi^2] \frac{z\sin\pi z}{2\pi} + 4\left[\frac{\sin\pi z}{\pi z} + \frac{z\pi\sin\pi z}{3!}\right] + \sum_{\text{even } n}{}' \frac{\sin\pi z \frac{4z^2}{(1 - n^2)}}{n^2\pi(z - n)}$$

Note the series are $O(\frac{1}{n^3})$, $O(\frac{1}{n^4})$, $O(\frac{1}{n^5})$ convergent respectively. However, large values of z require a "cancellation" convergence - the subtraction of large quantities from each other - which can get sticky. We will be operating, however, with z visible, so given accurate measurement this fact is of little import.

Needless to say, one can now rework the solution of the truncated

linear sets previously presented using these "faster" sampling series. (A typical example is discussed in Appendix II). The analytic arguments are paralleled by those previously given and we omit the details. It is easy to see that the double-alternant form is preserved in the "faster" equation set and, it is possible to consider the behaviour of the exact solution. In passing we note that these "faster" formulae for band-limited functions may be used in the previous integral expression for the mutual coupling to derive a rapidly convergent expression for the coupling involving only the sample points. Consider the expression for $g(z)$ $[-\pi,\pi]$ with $g'(0) = 0$ (maximum at $0° = \theta$) as an example. (We use the second expression requiring knowledge of the first derivative)

$$\frac{4\pi}{\zeta\beta} Z_{21} = \int_{-\infty}^{\infty} \frac{g(u)e^{-iuD}}{\sqrt{\beta^2 - u^2}} du = \int_{-\infty}^{\infty} \frac{\left[\frac{g(0)\sin\pi u}{\pi u} + \sum_{\mathbb{R}}' \frac{(-)^n(\sin\pi u)u \cdot g(n)}{\pi n(u-n)}\right]e^{-iuD}du}{\sqrt{\beta^2 - u^2}}$$

The first integral has been evaluated as a convolution. The remaining may be easily found with the following

$$\frac{\partial I(n,\beta,D)}{\partial D} = \int_{-\infty}^{\infty} - i \cdot u \cdot e^{-iuD} \cdot \frac{\sin\pi(u-n)}{\pi u - n\pi} \frac{du}{\sqrt{\beta^2 - u^2}}$$

which is directly proportional to the required quantity. (Note powers: $u^n \longleftrightarrow \delta^{(n)}(x)$ etc.). The numerical evaluation of $\partial_D I$ is simple, just the introduction of a $\cot\theta$ term in the integrand of the $1^{\underline{d}}$ form of $I(n,\beta,D)$ (cf. p. 7). So

$$\frac{4\pi}{\zeta\beta} Z_{21} = g(0) \sum_{\mathbb{R}} \frac{\text{sinc } u}{\sqrt{\beta^2 - u^2}} du + \sum_{\mathbb{N}}' \int_{\mathbb{R}} \frac{\text{sinc}(u-n)e^{iuD} \cdot udu \cdot (\frac{g(n)}{n})}{\sqrt{\beta^2 - u^2}}$$

$$= g(0)I(0,\beta,D) + \sum_{\mathbb{N}}' \frac{g(n)}{n} i \left[\frac{\partial I(n,\beta,D)}{\partial D}\right]$$

where the series convergence is now $\frac{1}{n^3}$ versus $\frac{1}{n^2}$ previously. That $\partial_D I_D \sim \frac{1}{n}$ is obvious from the convolution form of $I(n,\beta,D)$ over a finite interval. i.e. the Fourier coefficients of the $H_0^{(2)}$'s first derivative.

These arguments, again, may be made quite general with the accurate knowledge of each successive derivative of $g(u)$ at a point enhancing the rate of convergence of Z_{21} by an order of magnitude as above. Thus the series form of Z_{21} is made practicable if it is possible to exprapolate the first few values of $P(\frac{n\pi}{2a})$ outside the visible region. A perusal of the appendices hints that while least square "ball-park" results are readily forthcoming much work needs yet be done. The stabilization of an ill-posed problem, described heretofore in totally analytic terms, e.g. simultaneous solution of linear equations, may be accomplished if constraints exterior to the data are introduced. Now we leave the discussion of the "sampling-proximates" solutions to explore the general methods of Miller-Tikhonov regularization for such problems. We will later return to these truncated linear sets as subject for regularization.

Regularization is necessary when the problem at hand is "ill-posed" which is not to say unphysical. Required in this context may be 1) numerical differentiation 2) analytic continuation 3) solution of Fredholm or Volterra equations of the first kind etc. Many proper physical situations, e.g. computation of atmospheric density profiles, nuclear scattering, inverse scattering, remote sensing, and image enhancements in optics give rise to models that are "ill-posed" or "conditioned". The problem at hand, numerical analytic continuation of bandlimited functions, may be seen as ill-conditioned with the aid of the Riemann-Lebesgue Lemma. In the Fredholm formulation, we wish to solve for $f(x)$, with $g(x) = \int_V f(y)e^{iyx}dy$ $g(x)$ for $x \in v$ given. Now e^{iyx} is a smooth kernel, thus this integral

operator does not have a bounded inverse. To see this, let $f_\epsilon(y) = \sin n_\epsilon y$, note that for n_ϵ big $\left|\int_V f_\epsilon(y)e^{iyx}dy\right|$ may be made as small as desired by the Riemann-Lebesgue Lemma. Hence $f(y)$ and $f(y) + f_\epsilon(y)$ are indistinguishable as "solutions" i.e. finite changes in $f(y)$ may produce infinitesimal changes in $g(x)$. However, if one restricts suitably the characteristics of the solution, $f(y)$, e.g. take $\int_V (f'(y))^2dy$ a minimum, one produces a restricted problem whose solution is unique, with the continuous dependence on data, $g(x)$, restored. To this end we consider three such formulations. It is not known as yet which of these procedures are optimal or feasible. This must be the subject of future numerical research. We can only outline the various methods and indicate our reasons for preference or otherwise.

Tikhonov regularization [18, 19, 20] may be readily applied to the collocative solution of the Fredholm equation of the first kind [21]. In such a solution, the required extrapolation is just a matter of quadrature i.e. given $g(z) = \int_V f(x)e^{izx}dx$ and given $f(x)$, $g(z)$ is "defined" by quadrature outside $V = [-\beta,\beta]$. The problem is the determination of $f(x)$. Let $Tf(x) = \int_V f(x)e^{izx}dx$ and take an $L^2(V)$ setting [21]. If g is the exact pattern and g_v the measured pattern, let $\|g_v - g\|_V < v$. Introduce the functional $M(f,g_v,\alpha) = \|Tf - g_v\|^2 + \alpha W(f)$ where $W(f) = C_0\|f\|^2 + C_1\|f_1\|^2$ and $\|h(x)\|^2 = \int_V h(x)\bar{h}(x)dx$, i.e. the standard $L^2(V)$ norm. α, the regularization parameter, is ≥ 0, with C_0, C_1 likewise. A theorem [18, Chap. II] of Tikhonov requires that for all $g_v \epsilon L^2(V)$ there exist a unique, continuous, differentiable $f_{\alpha v}$ minimizing M given α, g_v. This $f_{\alpha v}$ is seen to be a stable solution to the original problem in least squares sense with $f_{\alpha v} \to f$ as $\{\alpha, v\} \to 0$ [18]. The $\alpha W(f)$ acts as a sort of Lagrange multiplier, restricting

oscillatory or excessive behaviour of f, frequently α is quite small - providing "just a bit" of damping. For such an $f_{\alpha V}$, the first variation must vanish i.e.

$$\delta_f[M(f,g_V,\alpha)] = 0 \Longleftrightarrow [T^*T + \alpha(C_0 I - C_1 D^2)]f_{\alpha V} = T^* g_V$$

where $T^*f = \int_V e^{-izx} f(x)dx$ and

$$T^*Tf = \int_V \frac{2\sin[(x' - x)\beta]}{(x' - x)\beta} f(x')dx'$$

Note T^* is the adjoint on $\mathbb{L}^2(V)$, D the differentiation operator, I identity etc. Thus a formal solution is

$$f_{\alpha V} = L^{-1}T^*g_V \quad \text{where} \quad L = [T^*T + \alpha(C_0 I - C_1 D^2)]$$

Note that the choice of α has not been specified. Here some of the "ad hoc" flavour of this technique sets in. For specific values of $C_0 = 1$, $C_1 = 0$ [21] it can be shown that $\|f_{\alpha V} - f_\alpha\| \leq v/2\sqrt{\alpha}$ where f_α minimizes $M(f,g,\alpha)$ (noise-free smoothing). For $\|f_\alpha - f\| \leq \gamma$, $\gamma \leq \frac{\alpha}{\alpha + \lambda_N}\|f\|$ where λ_N is $\max_i |\lambda_i|$ for $T^*Tf = \lambda_i f$ with $\gamma \ll \|f\|$, $\alpha \leq \lambda_N\|f\|$. Hence with some knowledge of $\|f\|(\sim\int|\text{current}|^2$magnitude in "slot") one has an upper bound on α. A bound on $\|f_{\alpha V} - f\|$ may be calculated and [21,19] given, thus $\|f_{\alpha V} - f\| \leq v/2\sqrt{\alpha} + (\alpha/\alpha + \lambda_N)\|f\|$. If the self-impedance of the antenna producing the given pattern is known, one might pick α such that the pattern solution provides the best approximation to the measured self-impedance (to be evaluated by the integral forms involving visible and invisible patterns for self-impedance discussed previously.) In general, the selection of α is a "questionable" procedure subject to numerical experiment. In using the Fredholm approach to continuation, we are required to

solve for a quantity, f, which is then integrated with another kernel, providing the required extrapolation. One might expect that the reintegration would obliterate some of the error in solution. However, we are also finding the inverse of a transform, f, only to retransform it, a seemingly redundant procedure. In fact, it can be shown [22,23] that the error in finding f is subject to a logarithmic continuity, whereas the optimal extrapolation error for extracting $g(u)$, $u \in$ invisible, is subject to a Hölder continuity i.e. error in $|f - f_v| \sim (\ell n\ v)^{-1}$ as $g_v \xrightarrow[\text{visible}]{} g$ whereas $\underset{u\,\in\,\text{invisible region}}{|g(u) - g_v(u)| \sim (v)^{\beta}}$ as $g_v \xrightarrow[\text{visible}]{} g$ with $0 < \beta < 1$ for optimal methods in both regards. The hazards of logarithmic continuity need numerical investigation in this connection. An important benefit in the use of the Tikhonov scheme is its ease of implementation. Whereas "optimal" methods referred to above make use of the Prolate Spheroidal Harmonics, the procedure outlined here requires only simple quadratures and the equations in question are stable. The descretization of all operators with any Gaussian Quadrature scheme (to minimize required sampling of the visible pattern) presents no nuisance and the inversion of L presents no difficulty. (Also numerically cheap!) Details of numerical experiments [21,18] are also encouraging.

A more elaborate technique of regularization, due to Miller [24] and extended by Viano et al., [23, 25 see also the exhaustive references listed there] makes good use of the Prolate Spheroidal Harmonics to establish the analytic continuation of bandlimited functions. While quite elegant, this theory's use of P.S.H. condemns one, in trying to implement it, to a host of numerical nightmares. Even recent algorithms [26] for computation of P.S.H. eigenfunctions and eigenvalues require messy matrix bisection/Strum e.v. sequences procedures which allow no simple change or rescaling without

complete recomputation at great expense. The only virtue in these techniques is their "best-possible" nature. Some consideration shows the method instinctively no different from those suggested [1] in our first progress report. We consider briefly these results.

Let X denote the set of all permissible pattern functions (i.e. $f \in X$ is band-limited appropriately, $\int_{\mathbb{R}} |f|^2 du < \infty$ etc). For $f \in X$ let Af denote whatsoever is subject to measurement (the "perfect" data in $\mathbb{V}$) approximately. Take h as the measured data. Af and h are contained in the "data space" Y (in this case $\mathbb{L}^2(\mathbb{V})$). To stabilize the problem, one constrains f by "boundary" or "global" values. Let B be a linear operator. Bf provides some constraint operation on f. $Bf \in Z$ the "constraint" space. Then if f_o satisfies

$$\|Af_o - h\|_Y \leq \epsilon \qquad \text{(satisfies expected data fit)}$$

$$\|Bf_o\|_Z \leq E \qquad \text{(satisfies global bound)}$$

it may be shown f_o also satisfies (by Lagrangian arguments) $\|Af_o - h\|_Y^2 + (\frac{\epsilon}{E})^2 \|Bf_o\|_Z^2 \leq 2\epsilon^2$. This last equation may be minimized by solution of the normal equations to obtain an f which can be shown to be "optimally" fit to f_o independent of norm used to measure the error [see, e.g., 24]. If the criterion of error is $\langle \cdot \rangle_e$, it may be shown $\langle f - f_o \rangle_e < \sqrt{2}\, M(\epsilon,E)$, where $M(\epsilon,E)$ is defined as $\sup\{\langle x\rangle_e \mid x \in X, \|Ax\|_Y < \epsilon, \|Bx\|_Z < E\}$ - the "best possible stability estimate." Called as such since $M(\epsilon,E)$ gives the "size" of all f that satisfy the global and data bounds, for such (f_1,f_2), $\|A(f_1 - f_2)\|_Y < 2$, $\|B(f_1 - f_2)\| \leq 2E$ so that $\langle f_1 - f_2\rangle_e \leq 2M(\epsilon,E)$ [see 23,24] etc.

Whence we may take $f_\epsilon = [A^*A + (\frac{\epsilon}{E})^2 B^*B]^{-1}A^*h$ as a regularized solution whose error (in $\langle\ \rangle_e$) is "best possible" but for a factor of

$\sqrt{2}$. Numerous estimates may be fashioned in terms of the eigenfunctions and spectrum of A, A^*A, etc. for the various error norms [cf. 23] but these estimates generally require full knowledge of the spectral decomposition of the various operators - something not readily available here.

Our problem is placed in the above setting when one considers the problem of extrapolation of optical images (see [25] for physical details.) We require explicity that the object image is of bounded energy E in $\mathbb{L}^2[-1,1]$.

Let $x = x(t)$, $t \in \mathbb{R}$, be $\mathbb{L}^2(\mathbb{R})$, bandlimited such that the image, $x(t) = \int_{-c/2n\pi}^{c/2\pi} e^{2\pi i t\omega}\,\tilde{x}(\omega)d\omega$ and take Ax to be the restriction of x to the interval $[-1,1]$. (We assume that the images are known, in a system of unit magnification, over an interval equal to the support of the object, $[-1,1]$.) Take $h = Ax + z$, the image corrupted by measurement error. The problem to estimate $x(t)$ given $h(t)$. In this context we identify the following

1) $X \longleftrightarrow \{\mathbb{L}^2(\mathbb{R})\} \cup \{\text{Bandlimited } f\}$

2) $Y \longleftrightarrow$ functions of compact support $[-1,1] \sim \mathbb{L}^2(-1,1)$

3) $Ax(t) = x(t) \quad t \in [-1,1], = 0 \quad$ otherwise

4) $A^*y(t) = \int_{-1}^{1} \frac{\operatorname{sinc}(t-s)}{\pi(t-s)}\, y(s)ds \quad y \in Y$ etc.

It is possible to specify a stabilization constraint

$$x(t) = \int_{-1}^{1} \frac{\operatorname{sinc}(t-s)}{(t-s)\pi} v(s)ds \quad \text{where} \quad \int_{-1}^{1} |v(s)|^2 ds < E^2$$

(band-pass filter etc.)

with regards to optical object energy. With suitable normalization one can take $B = (A^*)^{-1}$ and after much dust get a "best" approximation

$$x(t) = \sum_{k=0}^{\infty} \frac{\lambda_k^{3/2}}{\lambda_k^2 + (\frac{\epsilon}{E})^2} h_k \psi_k(c,t)$$

with $h_k = (h, v_k)_Y = \frac{1}{\sqrt{\lambda_k}} \int_{-1}^{1} h(t)\psi_k(c,t)dt$ and ψ_k the appropriate P.S.H. This review is not designed to be complete - details may be found in [23,24]. Others have considered the use of P.S.H. to extrapolate [27] band-limited functions with constraints and achieved similar results. Though keenly analytic in nature, they all evolve as quite cumbersome due to the use of P.S.H. We therefore continue to ignore such schemes save as possibly benchmark measures for other proposals. It shoud be noted that the constraining operator B, data fit constants (ϵ, E) need not both be known [24] to use the Miller method. Through certain concavity properties it suffices to know just <u>one</u> of these quantities. The selection of B, the <u>linear</u> constraint operator, is quite open. Further study, elsewhere, would seem appropriate.

Let us note that knowledge of the self impedance of a given antenna does not qualify as a constraint in the sense given above. (This does not exclude its use as a Lagrangian constraint as we shall see - it simply will not allow us to use it to stabilize the ill-posed problem.) Specifically, the input impedance $Z_{in} \alpha \int_{R} \frac{P(u)du}{\sqrt{\beta^2 - u^2}}$ is a <u>functional of P(u)</u> unless we are allowed to vary β. But P(u) implicitly (especially in 3^d) contains β, hence rendering different frequency measurements, unless P(u) is β invariable, worthless. The question of what information may be had from <u>different</u> freqyebct measurements of the visible pattern and impedance is open. It would seem necessary to have, a priori, some notion of antenna geometry to pursue this

problem.

That self-impedance specification will not stabilize the pro[illegible] is seen in the previous $\mathbb{L}^2(\mathbb{R})$ context of the Miller example. That $< P(u), \frac{1}{\sqrt{\beta^2 - u^2}} > =$ Const merely implies the projection of $P(u)$ on the $\mathbb{L}^2(\mathbb{R})$ vector $\frac{1}{\sqrt{\beta^2 - u^2}}$ is fixed - this does not preclude wild oscillations as previously discussed over sets of finite measure as would, say some limit on the average value of $P'(u)$ (mean squared). This does not mean one ought discard this information but, that it is not enough hence, one can use it as another Lagrangian constraint. (It turns out that if one tries to continue the pattern function (not power) the impedance operator can be reformulated to be of service, $\int_{\mathbb{R}} \frac{|f|^2}{\sqrt{\beta^2 - u^2}} du$ places some constraints on $\int_{\mathbb{R}} |f|^2 du$- but this case will not admit useful measurement.)

The final method of regularization applies to the solution of linear equations with error in the "right hand side" (i.e. the specific measurements of $g(\frac{n\pi}{2a})$ + Error (n)) and the stabilization thereof, [28,29,18, Chap. III]. These results apply directly to the truncated linear sets considered in the text. The setup is as follows:

$$\text{given } \underset{\approx}{A}\underset{\sim}{f} = \underset{\sim}{g} + \underset{\sim}{\epsilon} \quad \text{and} \sum_i \epsilon_i \leq e^2 << 1$$

how can one vary $\underset{\sim}{\epsilon}$ (since $\underset{\approx}{A}$, $\underset{\sim}{g}$ are fixed this is equivalent to "vary f")to minimize the a quadratic constraint on $\underset{\sim}{f}$, $Q = \sum_{ij} h_{ij} f_i f_j = \underset{\sim}{f}^* \underset{\approx}{H} \underset{\sim}{f}$. There are two cases of interest:

1) there exist $\{\underset{\sim}{f}^{(i)}\}_i^*$ such that Q is minimized and $\| \underset{\approx}{A}\underset{\sim}{f} - \underset{\sim}{g} \|^2 < e^2$. In this case $\underset{\sim}{f}^{(i)}$ is a variational solution to $\delta Q = 0$ and $\underset{\approx}{A}\underset{\sim}{f} = \underset{\sim}{g} + \underset{\sim}{\epsilon}$ - which is unlikely - $\underset{\sim}{f}^{(i)}$

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*i = index of vectors.

is however a valid solution.

2) For Q such that a variational minimum does not occur within $\{f\}$ such that $\|\epsilon\|^2 \le e^2$, we know that the minimum must occur on the boundary of $\{f \mid (Af = g + \epsilon$ or $f = A^{-1}(g + \epsilon))$ and $\|\epsilon\| = e\}$ or simply for $\{f \mid \|\epsilon\| = e\}$.

Therefore we may use the method of Lagrange Multiplier(s) to solve the following equivalent problem:

Minimize $Q = f^*Hf$, subject to the constraint

$(Af - g)^*(Af - g) = \epsilon^*\epsilon \quad e^2$ i.e. minimize the functional

$f^*Hf + \gamma_o \epsilon^*\epsilon$.

In our context, we may require a second Lagrange Multiplier to take into account any knowledge of the self-impedance associated with the measured visible pattern. In general, one may also require $Z_{in} = f \cdot c$. Thus for $F(f) = f^*Hf + \gamma_o\epsilon^*\epsilon + \gamma_1 f^*c$ we require $\delta_f F(f) = 0$

i.e. $$\frac{\partial}{\partial f_k}\{[f^*A^*Af - g^*Af - f^*A^*g + g^*g]\gamma_o + f^*Hf + \gamma_1 f^*c\} = 0$$

or $$(2A^*Af - A^*g - A^*g)\gamma_o + 2Hf + \gamma_1 c = 0 \quad \text{i.e.}$$

$$(A^*A\gamma_o + H)f = \gamma_o A^*g - \frac{\gamma_1}{2}c$$

where γ_o γ_1 are to be selected to satisfy the discrepancy and impedance constraints. (In practice the second constraint is linear and it incorporates directly into the solution of the above equations; if A, f are $n \times n$, n etc. , we have, with the equation $Z_{in} = f \cdot c$ $n + 1$ linear equations in terms of the $n + 1$ unknowns $\{f_i\}_{i=1}^n$ and γ_1, using γ_o

"as a known" parameter which we vary to (experimentally) satisfy $\underset{\sim}{\epsilon} * \underset{\sim}{\epsilon} = e^2$). The application of this technique to the previous sampling equation set(s) is immediate and will be considered and coded as soon as time permits. A few remarks concerning the method are in order.

1) The quadratic constraining function Q is essentially arbitrary. Possible forms for $\underset{\approx}{H}$ are "minimum variance", first difference (tabular), and prior departure from given form, e.g. the asymptotic pattern form.* Which form or forms are suitable must, perforce, be found in an empirical fashion.

2) One can show that the variation of $\underset{\sim}{f}$ with γ_o is slow and the ensuing numerical satisfaction of $\underset{\sim}{\epsilon} * \underset{\sim}{\epsilon} = e^2$ is "easy" with convexity arguments.

3) The method is a "generalized" least squares technique no more difficult than the solution of the normal equations. That it is possible to satisfy and solve the Lagrangian constraints (no inconsistency) is not clear from the above arguments - we refer to [18] for a discussion of this. The method was one of the first tractable solutions to the numerical problem of the First Kind Fredholm Equation, [29, circa 1962].

Conclusions and Extensions.

We have presented the rudiments of a theory for mutual coupling between canonical minimum scattering antennae requiring knowledge of the total pattern function. The problem of extending the visible pattern function throughout the entire required domain was solved by introducing the constraint of band-limitation. Whereupon, a series form representation of the required coupling integrals, utilizing various extensions of the

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*note $g(f) = \int_{-\pi}^{\pi} e^{izx} f(x)dx = f(\pi)(2\pi)\dfrac{\cos\pi(z-\frac{1}{2})}{\pi(z)}$ see, e.g. [30], an "observable " quantity.

sampling theorem, was developed. Lastly, we have proposed several techniques that should allow the computation of the invisible pattern values from the available data. Several well known regularization techniques were drawn from the literature in the context of our problem. The present research must now shift emphasis to the numerical validation of the "sampling" forms of continuation and/or comparison to the Miller-Tikhonov method. The ill conditioned numerical problem at hand required an extensive investigation into possible analytic forms of solution and having accomplished this, we now hope to devote substantially more time to the actual computations required. The appendix confirms at least order of magnitude results are possible so, at worst, "reasonable" bounds on the couplings may be expected.

There are several points novel to our approach. Intrinsic is the restriction to coupling amongst like antennae. This is required by the difficulty of phase measurement. The problem of phase retrieval [e.g. 31] is well known in optics but usually requires, say for a Logarithmic Hilbert transform resolution, knowledge of the pattern magnitude and its zeros over a relatively large interval, information here unknown. It may be possible to model or bound phase effects on the coupling and extrapolation formulae previously obtained, whereby the generality of the analysis would be restored. However, save at the expense of accurate phase measurement, the problem remains. The most useful observation is the bandlimited nature of the pattern power function. This lone fact restores the accuracy needed to consider extrapolation seriously. With the hope of extending the techniques to multiple dimensional antennae, we have kept the sampling theorem's multidimensional character in mind [32]. These notions would hopefully allow analysis of measured couplings between planar apertures and the like to follow through, whereas the various integral formu-

lations (Fredholm) would not readily reformulate. Note the assumption of bandlimitation to a degree holds implicitly some knowledge of that degree. To the extent all physical systems behold finite measure of energy and geometry (explicit in $2\frac{d}{}$) suggestive of the bandlimit, one may be forgiven its accurate assumption. The sensitivity of this assumption to error is another matter. Lastly, the optics literature, in considering the problems of image-processing and restoration - likewise demanding extrapolation - may yet be of further service.

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Appendix A

Numerical Illustration of the First Variational Method.

Correct and generated solutions to the extrapolation problem for various measure are presented. For this and subsequent examples the following bandlimited power pattern, typical of a $\frac{d}{2}$ slot radiator, will be employed. Note the pattern is real and even, whereby many simplifications ensue.

$$P(u) = \frac{\sin^2(\frac{\pi u}{2})}{(\frac{\pi u}{2})^2}, \quad P(0) = 1, \; P''(0) = \frac{-\pi^2}{6}$$

$$P(u) = \int_{-\pi}^{\pi} \frac{1}{(\frac{\pi}{2})^2} \left[\frac{\pi}{4} - \frac{|x|}{4} \right] e^{-iux} dx$$

$$= \sum_{\text{odd } n}{}' \frac{(-)^n \sin\pi z}{\pi(z-n)} \left[\frac{4}{\pi^2 n^2}\right] + \frac{\sin\pi z}{\pi z} \qquad \text{(Sampling Series)}$$

$$= \sum_{\substack{\text{odd } n \\ n > 0}} \frac{8z\sin\pi z}{\pi^3} \frac{1}{(n^2 - z^2)n^2} + \frac{\sin\pi z}{\pi z}$$

(utilization of $a_n = a_{-n}$)

$$= \frac{z\sin\pi z}{2\pi}\left(\frac{-\pi^2}{6}\right) + \left(\frac{\sin\pi z}{\pi z} + \frac{z\pi\sin\pi z}{3!}\right) + \sum_{\text{odd } n}{}' \frac{\sin\pi z \cdot z^2 \cdot \frac{4(-)^n}{\pi^2 n^2}}{n^2 \cdot \pi(n-z)}$$

("fast sampling series"

$$= \sum_{\substack{\text{odd } n \\ n > 0}} \left(\frac{8z^3}{\pi^3}\right) \frac{\sin\pi z}{n^4(n^2 - z^2)} - \frac{z\pi\sin z}{12} + \left(\frac{\sin\pi z}{\pi z} + \frac{z\pi\sin\pi z}{3!}\right)$$

("fast" series with $a_n = a_{-n}$ constraint)

Taking an $\mathbb{L}^2(-\pi,\pi)$ setting, pick as the unknown proximate to be matched on $(-\beta,\beta)$ $A(u) = \sum_{n=-N}^{N} a_n \frac{\sin\pi(z-n)}{\pi(z-n)}$ and invoke the constraint that $a_n = a_{-n}$, split the range of summation to obtain

$$A(u) = a_o \frac{\sin\pi z}{\pi z} + \sum_{n=1}^{N} \frac{(-)^n 2z\sin\pi z}{\pi(z^2 - n^2)} a_n$$

where $a_o = 1 = P(0)$, which is known a priori. As previously discussed, the $\{a_i\}$ need minimize the functional

$$J(\{a_i\}) = \int_{-\beta}^{\beta} (A(u) - P(u))^2 du$$

subject to the constraint of self-impedance. To wit:

$$\int_{-\beta}^{\beta} \frac{\sin^2 \frac{\pi u}{2}}{(\frac{\pi u}{2})^2} \frac{du}{\sqrt{\beta^2 - u^2}} = C_1 \quad \text{(resistive, measured)}$$

$$\int_{|u| > \beta} \frac{\sin^2 \frac{\pi u}{2}}{(\frac{\pi u}{2})^2} \frac{du}{\sqrt{u^2 - \beta^2}} = C_2 \quad \text{(reactive, measured)}$$

The last constraints provide the first two of $N + 2$ linear equations for the unknowns $\{a_i\}$, γ_1, γ_2

$$C_1 = \int_{\beta}^{\beta} A(u) \frac{du}{\sqrt{\beta^2 - u^2}} ; \qquad C_2 = \int_{|u| > \beta} A(u) \frac{du}{\sqrt{u^2 - \beta^2}}$$

The rest follow from the minimization of

$$J = \int_{-\beta}^{\beta} (\omega(u) - \sum_{n=1}^{N} \frac{(-)^n 2u\sin\pi u}{\pi(u^2 - n^2)} a_n)^2 \, du$$

$$+ \gamma_1 \int_{-\beta}^{\beta} A(u) \frac{du}{\sqrt{\beta^2 - u^2}} + \gamma_2 \int_{|u|>\beta} A(u) \frac{du}{\sqrt{u^2 - \beta^2}} \cdot$$

$$\delta J = 0 \longleftrightarrow$$

$$- 2 \sum_{n=1}^{N} a_n \int_{-\beta}^{\beta} \frac{2u(-)^i \sin\pi u}{\pi(u^2 - i^2)} \frac{\sin\pi u \cdot 2u(-)^n}{\pi(u^2 - n^2)} du$$

$$+ \int_{-\beta}^{\beta} \frac{4 \cdot u \cdot \sin\pi u \cdot (-)^i \omega(u)}{\pi(u^2 - i^2)} du + \gamma_1 \int_{\beta}^{\beta} \frac{(-)^i 2u\sin\pi u}{\pi(u^2 - i^2)} \frac{du}{\sqrt{\beta^2 - u^2}}$$

$$+ \int_{|u|>\beta} \frac{(-)^i 2u\sin\pi u}{\pi(u^2 - i^2)} \frac{du}{\sqrt{u^2 - \beta^2}} = 0 \quad \text{for} \quad i = 1,\ldots,N.$$

where $\omega(u) = \dfrac{\sin^2 \frac{u\pi}{2}}{(\frac{u\pi}{2})^2} - \dfrac{\sin u}{\pi u}$, also even. Notice all integrands are even and all singularities are weak or removable. To that end, let $t = \sqrt{\beta^2 - u^2}$ or $\sqrt{u^2 - \beta^2}$ where appropriate, then

$$C_1 = 2 \int_0^{\beta} A(u) \frac{du}{\sqrt{\beta^2 - u^2}} = 2 \int_0^{\beta} \frac{\sin\pi u}{\pi u} \frac{du}{\sqrt{\beta^2 - u^2}} + \sum_{n=1}^{N} \frac{(-)^n 4}{\pi} \int_0^{\beta} \frac{\sin\pi \sqrt{\beta^2 - v^2}}{\beta^2 - v^2 - n^2} dv \, a_n,$$

$$C_2 = 2 \int_{\beta}^{\infty} \frac{\sin\pi u}{\pi u} \frac{du}{\sqrt{u^2 - \beta^2}} + \sum_{n=1}^{N} \frac{(-)^n 4}{\pi} a_n \int_{\beta}^{\infty} \frac{\sin\pi\sqrt{t^2 \;\; \beta^2}}{\beta^2 + t^2 - n^2} dv;$$

$$\frac{(-)^i 4}{\pi} \int_0^{\beta} \frac{u\sin\pi u \; \omega(u)}{u^2 - i^2} du - \frac{16}{\pi^2} \sum_{n=1}^{N} a_n (-)^{i+n} \int_0^{\beta} \frac{u^2 \sin^2 \pi u}{(u^2 - i^2)(u^2 - n^2)} du$$

$$+ \frac{\gamma_1 4(-)^i}{\pi} \int_0^{\beta} \frac{\sin\pi\sqrt{\beta^2 - v^2}}{\beta^2 - v^2 - i^2} dv + \frac{4\gamma_2 (-)^i}{\pi} \int_{\beta}^{\infty} \frac{\sin\pi\sqrt{t^2 + \beta^2}}{\beta^2 + t^2 - i^2} dt = 0$$

Appendix B. Pt. I.

The Solution of Least-Square Sampling Proximates. Slow and Fast.

For various values of (β, N) a least square and direct solution of the Harris-type extrapolation procedure is provided. Again, the even slot-like pattern is used. Note the order of magnitude "contour" agreement.

Sampling proximate:

$$P(u_i) - \frac{\sin \pi u_i}{\pi u_i} = \sum_{n=1}^{N} \frac{u_i \sin u_i \pi}{(n^2 - u_i^2)} a_n \quad i = 1, \ldots, N.$$

$u_i \in [-\beta, \beta]$

"Fast" Sampling Proximate:

$$P(u_i) - \left(\frac{\sin \pi u_i}{\pi\, u_i} + \frac{\pi\, u_i \sin \pi u_i}{6} \right) - \frac{\pi u_i \sin \pi u_i}{12} =$$

$$\sum_{n=1}^{N} \frac{\sin \pi u_i \cdot u_i^2 \cdot a_n}{n^2 (n - u_i)}$$

where $i = 1, \ldots, N$, $u_i \in [-\beta, \beta]$. Note the solution to $\underset{\approx}{A}^T \underset{\approx}{A} \underset{\sim}{x}' = \underset{\approx}{A}^T \underset{\sim}{b}$ is given (cf. p. 14 etc.).

a_n n=1,3,5,7 ; a_n=0.00 for n even

```
0.4052848E 00
0.4503104E-01
0.1621138E-01
0.8271120E-02
0.5003512E-02
0.3349462E-02
0.2398137E-02
```

(.5,2)

```
0.4688860E 00
0.2749589E 00
-0.7994665E-04
0.1517328E-05
```

(.5,3)

```
0.4102466E 00
0.1564335E 00
0.3620858E 00
0.1051074E-04
0.3393462E-08
```

(6.0,3)

```
0.4063841E 00
-0.9943509E-01
0.1369655E 00
-0.6163890E-01
-0.4243187E 01
```

(1.5,4)

```
0.4053108E 00
0.1364395E-01
0.2487881E 00
0.2970130E 00
-0.2763855E-04
-0.1122543E-06
```

(3.5,4)

```
0.5617621E 00
-0.1075880E 00
0.2428225E 00
-0.9326207E 00
0.1817300E 00
0.3916962E 01
```

(1.5,5)

```
0.4054387E 00
0.4152496E-01
0.5998248E 00
0.3376217E 00
-0.7604128E 00
-0.5911746E-04
-0.2786582E-07
```

(6.0,4)

```
0.4106582E 00
-0.9564265E-01
0.1403168E 00
-0.1171724E-01
-0.2333261E-01
-0.4170190E 01
```

(5.0,7)

```
0.4052426E 00
0.4292067E-04
0.4504684E-01
-0.4674947E-04
0.1620189E-01
-0.6338101E-02
0.1724888E-02
0.9194569E-04
-0.1250425E-03
```

(12.0,7)

```
0.4059834E 00
-0.9111203E-02
0.4647213E-01
-0.5750649E-01
0.6536454E-01
-0.4907059E-01
0.5881457E-01
-0.1003777E 00
-0.7090366E 01
```

Note the extrapolative capacity is limited in keeping with the error in integration (about .0001)

Whereby we have sanitized most of the integrands for computation. These are still a number of numerical problems 1) the integrals over an unbounded range have a slow tail i.e.

$$\text{Tail} \sim \int_0^{\infty} \frac{\sin(\pi\sqrt{t^2 + \beta^2})}{t^2 + B}\,dt = \int_0^{1/A} \frac{\sin\pi\sqrt{\frac{1}{\omega^2} + \beta^2}}{B\omega^2 + 1}\,d\omega$$

where $\omega = 1/t$ is used and the infinite integrand "essentially" vanishes near $\omega \sim 0$ by cancellation from the $\sin(1/\omega)$ term. 2) The integral

$$\int_0^{\beta} \frac{\sin\pi u}{\pi u} \frac{du}{\sqrt{\beta^2 - u^2}}$$ provides some nuisance at $t = \beta$ which

may be solved by the substitutions $\omega = \beta - u$, $w = t^2$.

Results of solution for some (β, N) pairs follow.

Appendix B Part I

"answer" for a_n both fast and slow series

$a_1 = -.258012$ $a_7 = -.005265$

$a_3 = -.028668$ $a_9 = -.003185$

$a_5 = -.010320$ $a_{even} = 0.000$

(6,3.948)

SLOW		FAST	
-0.2580128E 00	-0.2580140E 00	-0.2580127E 00	-0.7164913E 01
-0.7353470E-06	-0.3924775E-06	-0.2513528E-06	-0.1891137E 01
-0.2866534E-01	-0.2866308E-01	-0.2866671E-01	-0.8529455E 00
-0.1517708E-04	-0.1946036E-04	-0.6962468E-05	0.7985000E 00
-0.7252075E-02	-0.6857496E-02	-0.8685432E-02	0.1262873E 00
-0.1112130E-01	-0.1178291E-01	-0.7425167E-02	0.4933339E-01

(8,5.102)

SLOW		FAST	
-0.2580130E 00	-0.9535195E 00	-0.2580127E 00	-0.1381086E 02
-0.1182210E-06	0.7450211E-01	-0.1826829E-06	-0.3846864E 01
-0.2866782E-01	-0.5386807E-01	-0.2866732E-01	-0.2088017E 01
-0.2204160E-06	0.7361603E-01	-0.8395742E-06	-0.9281958E 00
-0.1032621E-01	0.5622908E-01	-0.1034309E-01	0.3923310E 00
-0.1187794E-02	0.5404865E-01	-0.4816629E-02	0.3068506E 00
0.4335605E-02	0.3323541E-01	0.3168106E-01	0.1227144E 00
-0.1564984E-01	0.2338109E-01	-0.5178045E-01	0.6211779E-01

(10,6.262)

SLOW		FAST	
-0.2580128E 00	-0.9654347E 00	0.4477288E 00	-0.2605753E 02
-0.8167177E-07	0.6141235E-01	0.6240565E 00	-0.7202843E 01
-0.2866791E-01	-0.6953979E-01	-0.4193331E 00	-0.3865221E 01
-0.5019750E-07	0.5190228E-01	-0.4866355E 00	-0.2298439E 01
-0.1032097E-01	0.1193275E-01	0.9101173E 00	-0.1336782E 01
0.7897220E-05	0.6744343E-01	0.2018598E 00	-0.1192684E 00
-0.4118439E-02	0.6423628E-01	-0.1240659E 01	0.8324163E 00
-0.1631076E-01	0.3751091E-01	0.4555402E 00	0.3014658E 00
0.4584112E-01	0.2639338E-01	0.1300947E 01	0.1522411E 00
-0.4166836E-01	0.1999903E-01	-0.1237689E 01	0.8832186E-01

(14,8.542)

SLOW		FAST	
0.4118392E 00	-0.9762821E 00	0.4477288E 00	-0.6165854E 02
0.2302275E 00	0.4997449E-01	0.6240565E 00	-0.1656136E 02
-0.7756341E-01	-0.8212346E-01	-0.4193331E 00	-0.8416985E 01
-0.2591614E-01	0.3724670E-01	-0.4866355E 00	-0.5252716E 01
0.1113788E 00	-0.6742518E-02	0.9101173E 00	-0.3705853E 01
0.1484834E-01	0.3865260E-01	0.2018598E 00	-0.2568227E 01
-0.5962918E-01	0.1958247E-01	-0.1240659E 01	-0.1617739E 01
0.3482465E-01	0.4920435E-01	0.4555402E 00	-0.4387035E 00
0.4999132E-01	0.7716304E-01	0.1300947E 01	0.2828966E 01
-0.3561845E-01	0.3847015E-01	-0.1237689E 01	0.7982393E 00
-0.1402134E-01	0.2716770E-01	-0.8670908E 00	0.4016893E 00
0.4617713E-01	0.2093470E-01	0.1931193E 01	0.2394780E 00
-0.2492035E-03	0.1686515E-01	-0.1337657E-01	0.1557204E 00
-0.3241427E-01	0.1397950E-01	-0.2208042E 01	0.1071125E 00

Appendix B, II.

$$g(z) = \int_{-\pi}^{\pi} e^{izt}|\sin t|\,dt = \frac{2(1+\cos\pi z)}{1-z^2}$$

$$g_1(z) = \sum_{-2N}^{2N} \frac{\sin\pi z}{\pi(z-n)}\, g(n) \qquad \text{(Sampling series proximate)}$$

$$g_2(z) = \sum_{-2N}^{2N} \frac{\sin\pi z\; 4z^2}{n^2\pi(z-n)(1-n^2)} + [4-\pi^2]\frac{z\sin\pi z}{\pi} + 4\left[\frac{\sin\pi z}{\pi z} + \frac{z\pi\sin\pi z}{3!}\right]$$

(2nd derivative proximate)

evaluated for $2N = 2, 12, 24, 36, 48$

at points $z = 14.0 + (.221333)*k$, $k = 1,2,3,4,5$

Note the improvement possible for even $g(z)$ as follows:

$$\sum_{-M}^{M} \frac{\sin\pi z}{(z-n)}\, g(n) = g(0) + \sum_{n=1}^{M} \frac{\sin\pi z}{(z-n)}\, g(n) + \sum_{n=1}^{M} \frac{\sin\pi z}{(z+n)}\, g(-n)$$

$$= g(0) + \sum_{n=1}^{M} \frac{\sin\pi z\; g(n)}{\pi}\left(\frac{2z}{z^2-n^2}\right) \quad \text{converging now } O\left(\frac{1}{n^2}\right)$$

It is possible to reduce our labor considerably for $g(n)$ even or odd thus.

ANSWER

-0.1756904E-01	
-0.1136048E-01	
-0.4740268E-02	
-0.5820845E-03	
-0.4895511E-03	
0.1834701E-01	0.1444149E 00
-0.1685422E-02	0.8106232E-04
-0.1798547E-01	-0.1764774E-01
-0.1768929E-01	-0.1755905E-01
-0.1761997E-01	-0.1754761E-01
NEXT K	
0.2778002E-01	0.2243681E 00
-0.1989841E-02	0.7362366E-03
-0.1201773E-01	-0.1151848E-01
-0.1154983E-01	-0.1137543E-01
-0.1144117E-01	-0.1135540E-01
NEXT K	
0.2423075E-01	0.2007780E 00
-0.1306236E-02	0.1142502E-02
-0.5337119E-02	-0.4887581E-02
-0.4911125E-02	-0.4752159E-02
-0.4812896E-02	-0.4734039E-02
NEXT K	
0.9681560E-02	0.8227921E-01
-0.3699139E-03	0.6351471E-03
-0.8303262E-03	-0.6475449E-03
-0.6526075E-03	-0.5903244E-03
-0.6120130E-03	-0.5817413E-03
NEXT K	
-0.8909047E-02	-0.7764530E-01
0.2153404E-03	-0.7429123E-03
-0.2517626E-03	-0.4310608E-03
-0.4224665E-03	-0.4873276E-03
-0.4611164E-03	-0.4949570E-03

DATE

FILMED

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